Advanced Numerical Analysis, MATH 269A
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Homework 3
Due on Wednesday, October 30, 2024.

Problem 1: *Asymptotic orders of Euler methods.* Consider the ODE

$$y' = -100(y - \sin(t)), \qquad y(0) = 1$$

for $t \in [0, 3]$. Implement explicit Euler and implicit Euler. Use the final step of the implicit Euler simulation with $N = 10^7$ as a proxy for the true value of $y(T)$. Numerically observe that

$$y(T) - y_N \sim Ch^p \qquad \text{as } N \to \infty.$$

(a) What are the estimated orders p of explicit and implicit Euler? What are the estimated values of the constant C?

(b) Consider an incorrect implementation of implicit Euler with the update

$$y_{n+1} = y_n + hf(t_n, y_{n+1})$$

for $n = 0, \dots, N-1$. What is the estimated order p? What is the estimated value of the constant C?

Problem 2: *Debugging through asymptotic order.* Consider the ODE

$$y' = 4t^2 \cos(y), \qquad y(0) = 0$$

for $t \in [0, 1]$. Recall that Heun's method has the form

$$y^{n+1} = y_n + \frac{h}{2}\big(f(t_n, y_n) + f(t_{n+1}, y_n + hf(t_n, y_n))\big) \qquad \text{for } n = 0, 1, \dots, N-1.$$

Consider an incorrect version of Heun's method:

$$y^{n+1} = y_n + hf(t_{n+1}, y_n + hf(t_n, y_n)) \qquad \text{for } n = 0, 1, \dots, N-1.$$

For both methods, plot

$$\log_2 \left| \frac{y^{(M/2)}(T) - y^{(M)}(T)}{y^{(M/4)}(T) - y^{(M/2)}(T)} \right|$$

as $M \to \infty$, where $y^{(m)}(T)$ denotes the output of the simulation at the final step (the m-th step) with $N = m$ total steps. How do the estimated orders of the two methods compare?

Remark. In optimization, this "incorrect" method is called the *extragradient method.*

Problem 3: *If Newton converges, the limit is a root.* Assume $f\colon \mathbb{R}^d \to \mathbb{R}^d$ is continuously differentiable. Consider the Newton iteration

$$x_{n+1} = x_n - \big(Df(x_n)\big)^{-1} f(x_n) \qquad \text{for } n = 0, 1, \dots.$$

Assume $Df(x_n)$ is invertible for $n = 0, 1, \dots$ so that the iterates are well defined. Assume $x_n \to x_\infty$ and $Df(x_\infty)$ is invertible. Show that $f(x_\infty) = 0$.

Problem 4: *Heun's region of absolute stability.* Show that Heun's method has the region of absolute stability

$$S = \big\{ z \in \mathbb{C} \,\big|\, |1 + z + \tfrac{1}{2} z^2| < 1 \big\}.$$

Clarification. Recall that Heun's method has the form

$$y^{n+1} = y_n + \frac{h}{2}\big(f(t_n, y_n) + f(t_{n+1}, y_n + hf(t_n, y_n))\big) \qquad \text{for } n = 0, 1, \dots, N-1.$$

Problem 5: *For linear ODEs, (difference of RK) = (RK on difference).* Let $\{x(t)\}_{t=0}^T$ and $\{y(t)\}_{t=0}^T$ be solutions to the same linear ODE with different initial conditions:

$$\begin{aligned} x'(t) &= c + Ax, \qquad & x(0) &= x_0 \\ y'(y) &= c + Ay, \qquad & y(0) &= y_0, \end{aligned}$$

where $c \in \mathbb{R}^d$ and $A \in \mathbb{R}^{d \times d}$. Let $\{x_n\}_{n=0}^N$ and $\{y_n\}_{n=0}^N$ be outputs of an RK method applied to the same linear ODE. Let $z_n = x_n - y_n$ for $n = 0, \dots, N$. Show that $\{z_n\}_{n=0}^N$ is the output of the same RK method applied to the ODE

$$z'(t) = Az, \qquad z(0) = x_0 - y_0.$$

Clarification. Let $z(t) = x(t) - y(t)$. By linearity, it is clear that

$$z'(t) = Az, \qquad z(0) = x_0 - y_0.$$

The question is whether the difference of the RK simulations $z_n = x_n - y_n$ is equal to RK applied to the difference of the ODEs $z'(t) = Az$.
Hint. Recall that an s-stage RK method on the ODE $y' = f(t, y)$ is defined by the update

$$\begin{aligned} y_{n+1} &= y_n + h \sum_{i=1}^{s} b_i k_i \\ k_i &= f\Big(t_n + c_i h, y_n + h \sum_{j=1}^{s} a_{ij} k_j\Big), \qquad \text{for } i = 1, \dots, s. \end{aligned}$$

Problem 6: *Small enough stepsize exists.* Recall that Euler and Heun's methods have regions of absolute stability

$$S = \left\{ z \in \mathbb{C} \,\middle|\, |1+z| < 1 \right\}$$

and

$$S = \left\{ z \in \mathbb{C} \,\middle|\, |1+z+\tfrac{1}{2}z^2| < 1 \right\}.$$

Let $\lambda_i \in \{z \in \mathbb{C} \,|\, \mathrm{Re}(z) \neq 0\}$ for $i = 1, \ldots, d$. Show that there is a small enough $h > 0$ such that

$$h\lambda_i \in S \quad \text{if } \mathrm{Re}(\lambda_i) < 0 \qquad \textit{and} \qquad h\lambda_i \in \overline{S}^C \quad \text{if } \mathrm{Re}(\lambda_i) > 0$$

for $i = 1, \ldots, d$